

Properties of nonreciprocal light propagation in a nonlinear optical isolator

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Light propagation in a nonlinear optical medium is nonreciprocal for spatially asymmetric linear permittivity. We here examine physical mechanism and properties of such nonreciprocity (NR). For this, we calculate transmission of light through a two-level atom asymmetrically coupled to light inside open waveguides. We determine the critical intensity of incident light for maximum NR and a dependence of the corresponding NR on asymmetry in the coupling. Surprisingly, we find that it is mainly coherent elastic scattering compared to incoherent scattering of incident light which causes maximum NR near the critical intensity. We also show a higher NR of an incident light in the presence of an additional weak light at the opposite port.

Light propagation is nonreciprocal when transmission of light is different under reversal of incoming light's direction. Nonreciprocity (NR) in light propagation can be achieved using various physical mechanisms including magneto-optical Faraday rotation [1, 2], parametric modulations [3–6], optical nonlinearity plus spatially asymmetric linear permittivity [7–12] and spin-orbit interaction of light [13–15]. Optical NR without magnetic materials and fields has attracted a lot of interests in the recent years for its suitability in an on-chip integration of an optical isolator [3–22].

A few years ago, we proposed an all-optical diode or isolator [7] in a simple system consisting of a two-level atom (2LA) being asymmetrically coupled to light inside one-dimensional waveguides, such as superconducting transmission lines [23, 24] and line-defects in photonic crystals [25]. A propagating light inside such open waveguides can be tightly confined to deeply subwavelength sizes in the transverse dimensions. It leads to an effective photon-photon correlation through strong atom-photon coupling even at a lower light power [26]. The NR in the transmission of light in the proposed diode is achieved via (a) optical nonlinearity which results in an incoming light's power-dependent dielectric response of the system and (b) asymmetric coupling which creates a spatially asymmetric linear permittivity across the atom. Asymmetric permittivity causes a spatially asymmetric dielectric response. We [7] have shown that while single-photon transmission is the same under reversal of incoming light's direction but the two-photon transmission is not. This mechanism has been investigated in many recent studies [8–10, 12] for nonreciprocal transmission.

The proposed all-optical diode can be implemented in experiments with superconducting transmission lines coupled to an artificial atom, such as superconducting qubits [23, 24], or line-defects in photonic crystals coupled to quantum dots [25]. However, some significant modifications in the original calculation are required for an adequate description of these experimental systems. These are (a) incident light in coherent states instead of in Fock states and (b) incorporation of pure-dephasing and non-radiative photon loss from the atom either of which is inevitable in such physical systems. In this paper, we address these tasks for the above prototypical

model of a single asymmetrically-coupled 2LA [7] using quantum Langevin equations and Green's function method [27, 28] which is a bit similar [29] to the popular input-output theory or Heisenberg-Langevin equations approach [30–38]. Here we show that the NR in transmission depends nonmonotonically on the intensity of incoming light and asymmetry in the coupling. We calculate the critical intensity for which NR is maximum and also find a dependence of this maximum NR on asymmetry in the coupling. To our surprise, we find that while incoherent scattering has a larger contribution in NR at a higher intensity, it is mainly due to coherent elastic scattering of the incident light at a lower power. Finally, we show that NR of an incident light can be improved in the presence of an additional weak light at the opposite port.

We consider a 2LA with a transition frequency ω_e being direct-coupled to light at the left and right side of it with coupling strength g_L and g_R respectively. The Hamiltonian of the full system is

$$\frac{\mathcal{H}}{\hbar} = \omega_e \sigma^\dagger \sigma + \int_{-\infty}^{\infty} dk [v_g k (a_k^\dagger a_k + b_k^\dagger b_k + c_k^\dagger c_k + d_k^\dagger d_k) + (\sigma^\dagger (g_L a_k + g_R b_k + \gamma d_k) + \text{h.c.}) + \lambda \sigma^\dagger \sigma (c_k^\dagger + c_k)], \quad (1)$$

where we assume a linear energy-momentum dispersion for different photon modes with a group velocity v_g and write light-matter interactions in linear form within the rotating-wave approximation. Here σ^\dagger (σ) is the raising (lowering) operator of the 2LA, and a_k^\dagger , b_k^\dagger create a photon with wave number k respectively at the left and right side of the 2LA. The operators c_k^\dagger , d_k^\dagger respectively denote creation of excitations related to pure-dephasing (dominant in superconducting circuits) and non-radiative loss (dominant in atomic systems). The couplings λ and γ control strength of pure-dephasing and non-radiative loss. All the couplings are taken constant over photon frequency near ω_e ; this is known as the Markov approximation causing the photon fields to behave as memoryless baths. We also consider here that the couplings are turned on at $t = t_0$ when a light beam is shined on the 2LA.

We start the calculation by writing Heisenberg equations of motion for operators a_k , b_k , c_k , d_k , σ and $\sigma^\dagger \sigma$ appearing in the Hamiltonian in Eq. 1. These equations for

the photon operators a_k, b_k, c_k, d_k are first-order linear inhomogeneous differential equations which we solve formally for some initial condition at t_0 . The initial condition of photon operators indicates a direction of incoming photons. We get time-evolution of the photon operators, for example, $a_k(t)$ with an initial condition $a_k(t_0)$ as

$$a_k(t) = G_k(t - t_0)a_k(t_0) - ig_L \int_{t_0}^t dt' G_k(t - t')\sigma(t') \quad (2)$$

with $G_k(\tau) = e^{-iv_g k \tau}$, and similarly for $b_k(t), c_k(t), d_k(t)$. Plugging these solutions of the photon operators in the Heisenberg equations of the atomic operators σ and σ^\dagger , we find the following equations:

$$\begin{aligned} \frac{d\sigma}{dt} &= -i(\omega_e - i\Gamma_t)\sigma - i(1 - 2\sigma^\dagger\sigma)\eta_d(t) \\ &\quad - i\lambda(\sigma\eta_c(t) + \eta_c^\dagger(t)\sigma), \\ \frac{d\sigma^\dagger\sigma}{dt} &= -2\Gamma_d\sigma^\dagger\sigma + i\eta_d^\dagger(t)\sigma - i\sigma^\dagger\eta_d(t), \end{aligned} \quad (3)$$

where we identify $\eta_d(t) = \int_{-\infty}^{\infty} dk G_k(t - t_0)(g_L a_k(t_0) + g_R b_k(t_0) + \gamma d_k(t_0))$ and $\eta_c(t) = \int_{-\infty}^{\infty} dk G_k(t - t_0)c_k(t_0)$ as noises whose properties are determined by the initial condition of the photon fields at $t = t_0$. The rates $\Gamma_d = \Gamma_L + \Gamma_R + \Gamma_\gamma$ and $\Gamma_t = \Gamma_d + \Gamma_\lambda$ with $\Gamma_L = \pi g_L^2/v_g$, $\Gamma_R = \pi g_R^2/v_g$, $\Gamma_\gamma = \pi \gamma^2/v_g$, $\Gamma_\lambda = \pi \lambda^2/v_g$ denote dissipation and dephasing of the 2LA. The Eqs. 3,4 are in the form of quantum Langevin equations. The Langevin equations 3,4 are nonlinear differential equations of operators and have multiplicative noises.

The transmission and reflection coefficients and the non-radiative loss of scattered photons are calculated using a continuity equation, [27, 28]

$$\frac{d\sigma^\dagger\sigma}{dt} + \nabla \cdot j_p = 0, \quad (5)$$

where j_p is an operator for photon current. For an incident light from the left of the 2LA, we write $\nabla \cdot j_p = j_{pb} + j_{pd} - j_{pa}$, where j_{pa} and j_{pb} are photon current respectively at the left and right side of the 2LA, and j_{pd} is current of non-radiative photon loss. We find these current operators by plugging the Heisenberg equation for $\frac{d\sigma^\dagger\sigma}{dt}$ in Eq. 5:

$$j_{pa}(t) = ig_L \int_{-\infty}^{\infty} dk (a_k^\dagger(t)\sigma(t) - \sigma^\dagger(t)a_k(t)), \quad (6)$$

$$j_{pb}(t) = -ig_R \int_{-\infty}^{\infty} dk (b_k^\dagger(t)\sigma(t) - \sigma^\dagger(t)b_k(t)), \quad (7)$$

$$j_{pd}(t) = -i\lambda \int_{-\infty}^{\infty} dk (d_k^\dagger(t)\sigma(t) - \sigma^\dagger(t)d_k(t)). \quad (8)$$

At steady-state, $\frac{d\sigma^\dagger\sigma}{dt} = 0$ which results in $j_{pa} = j_{pb} + j_{pd}$. The transmission and reflection coefficients of light are respectively $j_{pb}/(v_g I_{in})$ and $1 - j_{pa}/(v_g I_{in})$ where I_{in} is the intensity of the incident light.

In this paper, we consider two different initial conditions of incoming light: (a) a single light beam from one

side of the 2LA and (b) two light beams from opposite sides of the 2LA.

(a) First, we consider a single input light in a coherent state $|E_p, \omega_p\rangle$ with a frequency ω_p and an amplitude E_p . We take everywhere the amplitude of light to be real for simplicity. For an input light from the left of the 2LA, we have $a_k(t_0)|E_p, \omega_p\rangle = E_p \delta(v_g k - \omega_p)|E_p, \omega_p\rangle$ and $b_k(t_0)|E_p, \omega_p\rangle = c_k(t_0)|E_p, \omega_p\rangle = d_k(t_0)|E_p, \omega_p\rangle = 0$. Thus, we get

$$I_{in} = \langle E_p, \omega_p | \int dk a_k^\dagger(t_0)a_k(t_0) | E_p, \omega_p \rangle = \frac{E_p^2}{2\pi v_g^2}, \quad (9)$$

where we use $\delta(k = 0) = (2\pi)^{-1}$.

We apply the above properties of coherent state to solve the nonlinear operator equations 3,4. By performing expectation of these operator-equations 3,4 in the initial state $|E_p, \omega_p\rangle$, we get rid of the noise operators. We define [33]

$$\mathcal{S}_1(t) = \langle E_p, \omega_p | \sigma(t) | E_p, \omega_p \rangle e^{i\omega_p(t-t_0)}, \quad (10)$$

$$\mathcal{S}_2(t) = \langle E_p, \omega_p | \sigma^\dagger(t)\sigma(t) | E_p, \omega_p \rangle, \quad (11)$$

and $\mathcal{S}_1^*(t) = (\mathcal{S}_1(t))^*$ which satisfy a closed set of linear coupled differential equations obtained from Eqs. 3,4. We write these equations in a compact manner by introducing vectors $\mathcal{S} = (\mathcal{S}_1(t), \mathcal{S}_1^*(t), \mathcal{S}_2(t))^T$ and $\mathcal{Q} = (-i\Omega_L, i\Omega_L, 0)^T$:

$$\frac{d\mathcal{S}}{dt} = \begin{pmatrix} i\delta\omega_p - \Gamma_t & 0 & 2i\Omega_L \\ 0 & -i\delta\omega_p - \Gamma_t & -2i\Omega_L \\ i\Omega_L & -i\Omega_L & -2\Gamma_d \end{pmatrix} \mathcal{S} + \mathcal{Q} \quad (12)$$

with detuning $\delta\omega_p = \omega_p - \omega_e$ and Rabi frequency $\Omega_L = g_L E_p/v_g$ for an incident light from the left of the 2LA. The Eq. 12 for such non-operator variables can be solved with an initial condition, e.g., $\mathcal{S}_1(t = t_0) = \mathcal{S}_1^*(t = t_0) = \mathcal{S}_2(t = t_0) = 0$ which indicates the 2LA in the ground state before shining a light on it. The long-time steady-state behavior of the system is independent of the initial condition for the 2LA. The steady-state solutions are obtained by setting $\frac{d\mathcal{S}}{dt} = 0$. These are

$$\mathcal{S}_1(t \rightarrow \infty) = \mathcal{S}_1(\infty) = \frac{-i\Omega_L(i\delta\omega_p + \Gamma_t)\Gamma_d}{\Lambda_L}, \quad (13)$$

$$\mathcal{S}_2(t \rightarrow \infty) = \mathcal{S}_2(\infty) = \frac{\Gamma_t \Omega_L^2}{\Lambda_L}, \quad (14)$$

where $\Lambda_L = \Xi + 2\Gamma_t \Omega_L^2$ with $\Xi = \Gamma_d(\Gamma_t^2 + \delta\omega_p^2)$.

Using the above solution of $\mathcal{S}_1(\infty)$ and $\mathcal{S}_2(\infty)$ we evaluate expectation value of the steady-state current operators, j_{pa}, j_{pb}, j_{pd} in the initial state $|E_p, \omega_p\rangle$. We denote $\langle E_p, \omega_p | j_{pa} | E_p, \omega_p \rangle$ by $\langle j_{pa} \rangle$ and so forth.

$$\begin{aligned} \langle j_{pa} \rangle &= -2(\Omega_L \text{Im}[\mathcal{S}_1(\infty)] + \Gamma_L \mathcal{S}_2(\infty)) \\ &= \frac{2\Omega_L^2 \Gamma_t (\Gamma_R + \Gamma_\gamma)}{\Lambda_L}, \end{aligned} \quad (15)$$

$$\langle j_{pb} \rangle = 2\Gamma_R \mathcal{S}_2(\infty) = \frac{2\Omega_L^2 \Gamma_t \Gamma_R}{\Lambda_L}, \quad (16)$$

$$\langle j_{pd} \rangle = 2\Gamma_\gamma \mathcal{S}_2(\infty) = \frac{2\Omega_L^2 \Gamma_t \Gamma_\gamma}{\Lambda_L}. \quad (17)$$

Indeed $\langle j_{pa} \rangle = \langle j_{pb} \rangle + \langle j_{pd} \rangle$ in the steady-state. The transmission coefficient of light from left to right side of the 2LA is $\mathcal{T}_{LR} = \langle j_{pb} \rangle / (v_g I_{\text{in}}) = 4\Gamma_t \Gamma_L \Gamma_R / \Lambda_L$. \mathcal{T}_{LR} depends asymmetrically on Γ_L and Γ_R due to the term $2\Gamma_t \Omega_L^2$ in the denominator Λ_L . The term $2\Gamma_t \Omega_L^2$ is related to the intensity of incident light from the left. Transmission coefficient \mathcal{T}_{RL} from right to left of the 2LA is found by exchanging Γ_L and Γ_R in \mathcal{T}_{LR} . Thus, a difference in light transmission under reversal of incident light is

$$\Delta\mathcal{T} = \mathcal{T}_{LR} - \mathcal{T}_{RL} = 4\Gamma_t \Gamma_L \Gamma_R \left(\frac{1}{\Lambda_L} - \frac{1}{\Lambda_R} \right), \quad (18)$$

with $\Lambda_R = \Xi + 2\Gamma_t \Omega_R^2$. The transmission difference $\Delta\mathcal{T}$ is a measure of NR in this system. It vanishes when $g_L = g_R$. It also vanishes in the single-photon limit of the incident light [7] for $E_p \rightarrow 0$ when the terms $2\Gamma_t \Omega_L^2$ in Λ_L and $2\Gamma_t \Omega_R^2$ in Λ_R are dropped. Therefore, both the asymmetry in coupling and optical nonlinearity at higher light intensity are essential for nonreciprocal transmission of light in the current system.

We plot lineshape of \mathcal{T}_{LR} , \mathcal{T}_{RL} and $\Delta\mathcal{T}$ with a scaled intensity of incident light in Fig. 1(a) for $\Gamma_L/\omega_e = 0.1$ and $\Gamma_R/\omega_e = 0.3$. Both \mathcal{T}_{LR} , \mathcal{T}_{RL} fall monotonically with increasing I_{in} due to photon-blockade in a direct-coupled system. A 2LA is saturated by a single photon; therefore it acts as a nonlinear medium for two or multiple photons. However, the strength of such optical nonlinearity by a single 2LA is expected to fall above a critical photon number as multiple photons can not simultaneously interact with a 2LA. Consequently, nonreciprocity in light transmission through a 2LA would decrease above a critical intensity of the incident light. Indeed we find from Eq. 18 that $\Delta\mathcal{T}$ increases with I_{in} up to a critical value $I_{\text{in}}^{\text{cr}} = \Xi / (4v_g \Gamma_t \sqrt{\Gamma_L \Gamma_R})$ before falling monotonically with a further increase in I_{in} [12]. The nonmonotonic nature of $\Delta\mathcal{T}$ with increasing I_{in} is shown in Fig. 1(a,b) for $\Gamma_L/\omega_e = 0.1$ and different $\Gamma_R/\omega_e = 0.2, 0.3, 0.4$. In Fig. 1(a) we also plot a scaled transmission difference $\Delta\mathcal{T}/\bar{\mathcal{T}} \equiv 2\Delta\mathcal{T}/(\mathcal{T}_{LR} + \mathcal{T}_{RL})$ which increases monotonically with increasing I_{in} before saturating at large I_{in} . Interestingly, $\Delta\mathcal{T}/\bar{\mathcal{T}}$ at high I_{in} shows a large value where light transmission through the 2LA itself is bit small.

For fixed Γ_L and Γ_R , the maximum NR, $\Delta\mathcal{T}^{\text{cr}}$ is achieved at $I_{\text{in}} = I_{\text{in}}^{\text{cr}}$,

$$\Delta\mathcal{T}^{\text{cr}} = \frac{4\Gamma_t \Gamma_L \Gamma_R (\Gamma_R - \Gamma_L)}{\Xi (\Gamma_R + \Gamma_L + 2\sqrt{\Gamma_L \Gamma_R})}. \quad (19)$$

The dependence of $\Delta\mathcal{T}^{\text{cr}}$ on asymmetry in the coupling is nontrivial which can be seen by plotting $\Delta\mathcal{T}^{\text{cr}}$ with Γ_L and Γ_R . Instead, we plot $\Delta\mathcal{T}^{\text{cr}}$ with Γ_R/ω_e in Fig. 1(c) for a fixed $\Gamma_L/\omega_e (= 0.1)$. $\Delta\mathcal{T}^{\text{cr}}$ changes nonmonotonically with Γ_R and its sign switches across $\Gamma_R = \Gamma_L$. For a fixed Γ_L , $\Delta\mathcal{T}^{\text{cr}}$ becomes extreme at two values of Γ_R – one value is smaller than Γ_L , and another is larger than Γ_L . In Fig. 1(c) we also show how light detuning $\delta\omega_p$, pure dephasing Γ_λ , and non-radiative loss Γ_γ affect NR. We find while a smaller $\delta\omega_p$ or Γ_γ increases the magnitude of $\Delta\mathcal{T}^{\text{cr}}$ significantly, Γ_λ has relatively less influence

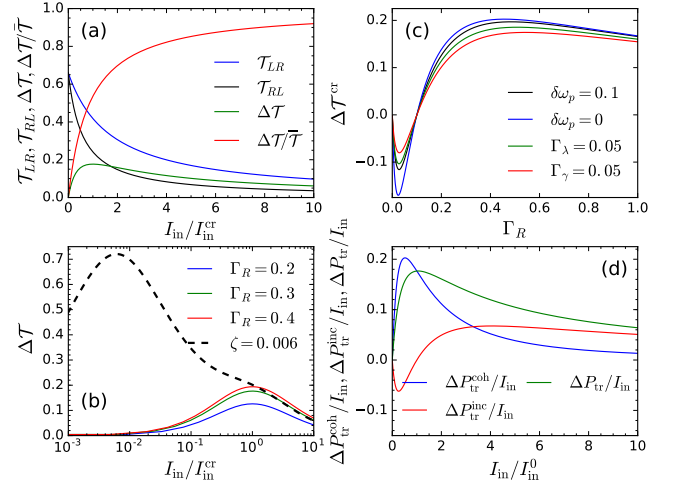


FIG. 1. Features of nonreciprocal transmission. (a) Transmission coefficients \mathcal{T}_{LR} , \mathcal{T}_{RL} , a difference in transmission $\Delta\mathcal{T}$ and a normalized difference in transmission $\Delta\mathcal{T}/\bar{\mathcal{T}}$ with scaled intensity $I_{\text{in}}/I_{\text{in}}^{\text{cr}}$ of incident light. (b) $\Delta\mathcal{T}$ vs. $I_{\text{in}}/I_{\text{in}}^{\text{cr}}$ at $\Gamma_L = 0.1$ and three different Γ_R . The dashed curve is for $\Gamma_L = 0.1$ and $\Gamma_R = 0.3$ in the presence of an additional backward light of intensity I_b with $\zeta = I_b/I_{\text{in}}^{\text{cr}} = 0.006$. (c) A difference in transmission $\Delta\mathcal{T}^{\text{cr}}$ at the critical intensity $I_{\text{in}}^{\text{cr}}$ with Γ_R for a fixed $\Gamma_L = 0.1$ and different values of Γ_γ , Γ_λ and $\delta\omega_p$. (d) Contribution of coherent ($P_{\text{tr}}^{\text{coh}}$) and incoherent ($P_{\text{tr}}^{\text{inc}}$) scattering in nonreciprocity. In all the plots $\Gamma_L = 0.1$, $\Gamma_R = 0.3$, $\Gamma_\gamma = \Gamma_\lambda = 0.01$ and $\delta\omega_p = 0.1$ if they are not explicitly mentioned. The rates Γ_L , Γ_R , Γ_γ , Γ_λ and $\delta\omega_p$ are in units of ω_e and $v_g = 1$.

on NR. We notice that the maximum value of $\Delta\mathcal{T}^{\text{cr}}$ for our studied parameters is around 0.2 which is comparatively small for practical applications.

To have a better understanding of the underlying physical mechanism of this NR, we now investigate a role of coherent and incoherent scattering of light in NR. Coherently scattered light has a constant phase relation with the incident light and can be detected using phase-sensitive homodyne-type measurements. We evaluate power spectrum of the transmitted light to find coherent and incoherent parts in it. For this, we introduce a real-space description of the photon operators at position $x \in [-\infty, \infty]$ of both sides of the 2LA. We define photon operator as $a_x(t) = \int_{-\infty}^{\infty} dk e^{ikx} a_k(t) / \sqrt{2\pi}$ and $b_x(t) = \int_{-\infty}^{\infty} dk e^{ikx} b_k(t) / \sqrt{2\pi}$ where the operators at $x < 0$ and $x > 0$ represent respectively incident and scattered photons on each side of the 2LA and the photons at $x = 0$ are coupled to the 2LA.

For an incident light from left of the 2LA, the power spectrum of transmitted light at long-time steady-state is defined as

$$P_{\text{tr}}(\omega) = \text{Re} \int_0^\infty \frac{d\tau}{\pi} e^{i\omega\tau} \langle b_x^\dagger(t) b_x(t+\tau) \rangle, \quad (20)$$

where we take $x > 0, t \gg t_0$ and the expectation $\langle \dots \rangle$ is again performed in the initial state $|E_p, \omega_p\rangle$. An expression like Eq. 20 for $a_x(t)$ at $x < 0, t < t_0$ would give a

power spectrum of the incident light, $P_{\text{in}}(\omega) = E_p^2 \delta(\omega - \omega_p) / (2\pi v_g^2)$. Thus, total incident power, $\int d\omega P_{\text{in}}(\omega) = I_{\text{in}}$. To calculate $P_{\text{tr}}(\omega)$, we first apply a formal solution of the Heisenberg equation for $b_k(t)$ like of Eq. 2, and rewrite $P_{\text{tr}}(\omega)$ using input fields and atomic operators. Applying $b_k(t_0)|E_p, \omega_p\rangle = 0$, we find

$$P_{\text{tr}}(\omega) = \frac{2\Gamma_R}{\pi v_g} \text{Re} \int_0^\infty d\tau e^{i\omega\tau} \langle \sigma^\dagger(t) \sigma(t+\tau) \rangle.$$

Thus, we now need to calculate a two-time correlation of atomic operators $\langle \sigma^\dagger(t) \sigma(t+\tau) \rangle$ to proceed further. So we define three new correlators [33]: $\mathcal{S}_3(\tau) = \langle \sigma^\dagger(t) \sigma(t+\tau) \rangle e^{i\omega_p\tau}$, $\mathcal{S}_4(\tau) = \langle \sigma^\dagger(t) \sigma^\dagger(t+\tau) \rangle e^{-i\omega_p(2(t-t_0)+\tau)}$, and $\mathcal{S}_5(\tau) = \langle \sigma^\dagger(t) \sigma^\dagger(t+\tau) \sigma(t+\tau) \rangle e^{-i\omega_p(t-t_0)}$, which are t -independent at long-time steady-state. Notice here, $\int d\omega P_{\text{tr}}(\omega)/I_{\text{in}} = 2\Gamma_R \langle \sigma^\dagger(t) \sigma(t) \rangle / (v_g I_{\text{in}}) = \mathcal{T}_{LR}$.

It can be shown after some algebra that $\mathcal{S}_3(\tau)$, $\mathcal{S}_4(\tau)$ and $\mathcal{S}_5(\tau)$ satisfy a set of inhomogeneous differential equations similar to those in Eq. 12 when \mathbf{S} and $\mathbf{\Omega}$ are replaced respectively by $\tilde{\mathbf{S}} = (\mathcal{S}_3(t), \mathcal{S}_4(t), \mathcal{S}_5(t))^T$ and $\tilde{\mathbf{\Omega}} = (-i\Omega_L \mathcal{S}_1^*(\infty), i\Omega_L \mathcal{S}_1^*(\infty), 0)^T$. The initial condition and asymptotic behavior in the limit $\tau \rightarrow \infty$ of these correlations are: $\mathcal{S}_3(\tau=0) = \mathcal{S}_2(\infty)$, $\mathcal{S}_4(\tau=0) = \mathcal{S}_4(\tau=0) = 0$, and $\mathcal{S}_3(\tau \rightarrow \infty) = |\mathcal{S}_1(\infty)|^2$, $\mathcal{S}_4(\tau \rightarrow \infty) = (\mathcal{S}_1^*(\infty))^2$, $\mathcal{S}_5(\tau \rightarrow \infty) = \mathcal{S}_1^*(\infty) \mathcal{S}_2(\infty)$. Using these long- τ limit, we now define $\delta\mathcal{S}_j(\tau) = \mathcal{S}_j(\tau) - \mathcal{S}_j(\tau \rightarrow \infty)$ with $j = 3, 4, 5$ which satisfy a set of homogeneous differential equations similar to the homogeneous part of Eq. 12. We solve these coupled differential equations to evaluate the power spectrum $P_{\text{tr}}(\omega) = P_{\text{tr}}^{\text{coh}}(\omega) + P_{\text{tr}}^{\text{inc}}(\omega)$ where $P_{\text{tr}}^{\text{coh}}(\omega)$ and $P_{\text{tr}}^{\text{inc}}(\omega)$ represent respectively coherent and incoherent parts of the transmitted power [39].

For an incoming light from the left of the 2LA, we calculate coherent and incoherent parts of the total transmitted power by taking integration over ω of $P_{\text{tr}}^{\text{coh}}(\omega)$ and $P_{\text{tr}}^{\text{inc}}(\omega)$ respectively. By switching Γ_L and Γ_R , we find coherent and incoherent parts of the total transmitted power for an incident light from the right of the 2LA. A difference in total coherent and incoherent transmitted power under reversal of incident light and after scaling by total incoming power I_{in} are:

$$\Delta P_{\text{tr}}^{\text{coh}}/I_{\text{in}} = 4\Gamma_L \Gamma_R \Gamma_d \Xi (\Lambda_L^{-2} - \Lambda_R^{-2}), \quad (21)$$

$$\Delta P_{\text{tr}}^{\text{inc}}/I_{\text{in}} = 4\Gamma_L \Gamma_R (\Gamma_\lambda \Xi (\Lambda_L^{-2} - \Lambda_R^{-2}) + 2\Gamma_t^2 (\Omega_L^2 \Lambda_L^{-2} - \Omega_R^2 \Lambda_R^{-2})). \quad (22)$$

As expected, we get $(\Delta P_{\text{tr}}^{\text{coh}} + \Delta P_{\text{tr}}^{\text{inc}})/I_{\text{in}} \equiv \Delta P_{\text{tr}}/I_{\text{in}} = \Delta\mathcal{T}$ of Eq. 18. $\Delta P_{\text{tr}}^{\text{inc}}/I_{\text{in}}$ goes through zero at a finite incident intensity I_{in}^0 where

$$I_{\text{in}}^0 = \frac{1}{2v_g} (-\rho_1 + \sqrt{\rho_1^2 - \rho_2}) \quad \text{with} \quad (23)$$

$$\rho_1 = \frac{\Xi \Gamma_\lambda (\Gamma_R + \Gamma_L)}{4\Gamma_t^2 \Gamma_R \Gamma_L}, \quad \rho_2 = \frac{\Xi^2 (\Gamma_\lambda^2 - \Gamma_d^2)}{4\Gamma_t^4 \Gamma_R \Gamma_L}.$$

We plot $\Delta P_{\text{tr}}^{\text{coh}}/I_{\text{in}}$, $\Delta P_{\text{tr}}^{\text{inc}}/I_{\text{in}}$ and $\Delta P_{\text{tr}}/I_{\text{in}}$ with a scaled intensity $I_{\text{in}}/I_{\text{in}}^0$ in Fig. 1(d). The Fig. 1(d) shows that

$\Delta P_{\text{tr}}^{\text{coh}}/I_{\text{in}}$ and $\Delta P_{\text{tr}}^{\text{inc}}/I_{\text{in}}$ have opposite sign for $I_{\text{in}} < I_{\text{in}}^0$ where they act against each other to reduce $\Delta P_{\text{tr}}/I_{\text{in}}$. Interestingly, the main contribution to NR at lower light power comes from coherently scattered light of a frequency that of the incident light. It indicates that the mixing of incident photon modes is not essential for NR [7]. NR at a higher light power is mainly due to incoherent scattering.

(b) Finally, we consider the presence of an additional small-amplitude backward light along with a large-amplitude forward light. Shi *et al.* [11] have recently studied this situation to show a decline in NR due to dynamic reciprocity in a nonlinear optical isolator for an additional backward light whose spectral band does not overlap with the forward light. We here investigate the other case when the spectral band of backward light overlaps with the forward light. In this case, we have initial condition $a_k(t_0)|\phi\rangle = E_p \delta(v_g k - \omega_p)|\phi\rangle$, $b_k(t_0)|\phi\rangle = E_b \delta(v_g k - \omega_b)|\phi\rangle$ and $c_k(t_0)|\phi\rangle = d_k(t_0)|\phi\rangle = 0$ for an initial state $|\phi\rangle = |E_p, \omega_p\rangle \otimes |E_b, \omega_b\rangle$. $|\phi\rangle$ is a product of the states of forward and backward light at the left and right side of the atom with respective frequency ω_p, ω_b and amplitude E_p, E_b . We are mostly interested in the regime when $E_b < E_p$ and we take $\omega_b = \omega_p$ for an overlap of the spectrum of monochromatic lights. We next find steady-state variables $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ in Eqs. 10, 11 for the initial state $|\phi\rangle$ and use them to calculate corresponding $\langle j_{pa} \rangle$ and $\langle j_{pb} \rangle$.

The total transmission coefficient \mathcal{T}_{LR} of light from left to right side of the 2LA is found by dividing $\langle j_{pb} \rangle / v_g$ by total intensity $(I_{\text{in}} + I_b)$ of forward and backward light where $I_b = E_b^2 / (2\pi v_g^2)$ is an intensity of backward light. As before, we get transmission coefficient \mathcal{T}_{RL} for a forward light from right to left of the 2LA by exchanging Γ_L and Γ_R in the above \mathcal{T}_{LR} . In Fig. 1(b), we add $\Delta\mathcal{T} = \mathcal{T}_{LR} - \mathcal{T}_{RL}$ with $I_{\text{in}}/I_{\text{in}}^{\text{cr}}$ of a forward light for $I_b/I_{\text{in}}^{\text{cr}} = 0.006$. We observe a significant enhancement in $\Delta\mathcal{T}$ at $I_{\text{in}} < I_{\text{in}}^{\text{cr}}$ due to the additional backward light, and $\Delta\mathcal{T}$ has a peak around $I_{\text{in}} \approx I_b$. The following argument can explain the increase in $\Delta\mathcal{T}$. Let us consider that our device facilitates transmission of a forward light from the left of the 2LA, and then it would also facilitate transmission of a backward light from the left for a forward light from the right of the 2LA. Thus, we have an overall improvement in $\Delta\mathcal{T}$ whenever $I_{\text{in}} \sim I_b$. However, a practical regime of light powers is $I_b < I_{\text{in}} < I_{\text{in}}^{\text{cr}}$ when the increase NR of a large-amplitude forward light due to a small-amplitude backward light may have some potential application.

In conclusion, we have shown several exciting features of nonreciprocal light transmission through a nonlinear optical isolator based on an exact microscopic analysis. Our theoretical analysis can readily be extended to more complex microscopic models of a nonlinear optical isolator such as in Refs. [8–10, 12, 40].

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